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Wave-breaking limits for nonquasistatic oscillations in a warm one-dimensional electron plasma

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In this work, wave breaking for general non-quasi-static oscillations in warm plasma is investigated using Lagrangian methods. In particular, the effects of secular behavior on wave-breaking limits are explored, and it is shown that thermal effects can sometimes prevent wave breaking by curbing secular behavior. The oscillation equations for fully relativistic warm plasma are cast into Lagrangian form, and wave-breaking limits are derived for waves in warm plasma having nonconstant density. These results have important applications in electron acceleration schemes that employ a wakefield or a slow beat wave propagating down a density gradient.

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The problem of wave breaking of one-dimensional (1D) quasistatic waves, i.e., waves that are nonevolving with respect to a comoving coordinate $\xi = x - v_\varphi t$ for some fixed speed v_φ, has been solved completely [1–6]. However, there are many configurations of nonquasistatic waves for which it is also important to know the wave-breaking limits. Wave breaking in plasmas having nonconstant background densities, for example, is of much importance to laser resonance absorption [7,8], electron acceleration schemes using colliding laser pulses on downward density slopes [9,10], and two-stage injection-acceleration schemes where electrons are injected into a laser-driven wakefield at high plasma density and subsequently accelerated at a lower density [11]. In a recent experiment [12], electron bunches having very low absolute longitudinal momentum spread (0.17 MeV/c) have been produced by wave breaking of a plasma wave propagating down a density gradient, as predicted by Bulanov *et al.* [13]. Furthermore, a plasma oscillation may have a spatially varying amplitude for any number of reasons, and this has its own peculiar effects on wave breaking if the oscillation is relativistic [14]. While various cases of breaking of nonquasistatic waves have been investigated for cold plasma [1,13–16], there are hardly any results for such waves in the presence of thermal effects. Since a realistic plasma needs to have a finite temperature to prevent recombination, it is important that the theory of nonquasistatic plasma oscillations is expanded to include thermal effects also.

The wave-breaking limits for quasistatic waves are well established. For a quasistatic wave with amplitude A, frequency ω, and wave number k, cold-plasma wave breaking sets in at $kA=1$ or $v=v_\varphi \equiv \omega/k$ [1,2]. Warm-plasma wave breaking occurs at $v=v_\varphi[1-\alpha k_B T/(m v_\varphi^2)^{1/(\alpha+1)}]$ for nonrelativistic plasma [3,4] [where $\alpha=3$ ($\alpha=1$) denotes adiabatic (isothermal) compression], or $v=v_\varphi[1-\sqrt{\beta/\gamma_\varphi^2+\sqrt{\beta}/(\gamma_\varphi^3 v_\varphi)}]/(1+v_\varphi\sqrt{\beta}/\gamma_\varphi)$ for relativistic plasma [5,6] [where $\beta=3k_B T/(m_e c^2)$ and $\gamma_\varphi^2=1/(1-v_\varphi^2)$]. For nonquasistatic waves however there are several controversies in the literature which must be resolved before one can proceed. First, there is the discrepancy between Dawson's limit $kA=1$ [1,17] and the Davidson-Schram limit $kA=1/2$ [18–21]. This is easily resolved by noting that Dawson's solution is written in Lagrangian coordinates $(\bar{x},\tau)$, where $\bar{x}$ denotes the average position of an oscillating electron, while the Davidson-Schram solution is written in Lagrangian coordinates (x_0,τ), where x_0 denotes the electron position at $\tau=0$. Using $\partial\bar{x}/\partial x_0=n(x_0,0)/n_0$ [22], where n_0 is the density of stationary ions and $n(x_0,0)$ denotes the electron density at $\tau=0$, it follows immediately that both solutions are in fact equivalent. The effective wave number k_{eff} is then defined as the increase in the electron phase with $\bar{x}$ for constant τ as the electron phase at $x(\bar{x},\tau)=\bar{x}$ is constant (so $\bar{x}$ provides a good reference point) while the phase at $x(x_0,\tau)=x_0$ is not. Then Dawson's solution yields $k_{\text{eff}}=k$, while the Davidson-Schram solution yields $k_{\text{eff}}=k/(1-kA)$, and wave breaking sets in at $k_{\text{eff}}A=1$ for both. This resolves the apparent conflict.

Of a more serious nature is the claim by Infeld and Rowlands [23] that the wave-breaking limit in thermal plasma in the absence of secular behavior (see below) is independent of the plasma temperature. This would contradict all the existing results on quasistatic waves [3–6] and would imply that quasistatic waves cannot be considered a special case of 1D plasma oscillations. However, a close scrutiny of Ref. [23] reveals that the wave solutions presented there do not match the nonlinear evolution equation for a nonrelativistic thermal plasma near the cold-plasma wave-breaking limit. In Dawson's coordinates $(\bar{x},\tau)$, this equation reads as [23]

$$\left(\frac{\partial^2}{\partial\tau^2}+\omega_p^2\right)\Psi=\omega_p^2-v_T^2\frac{\partial^2\Psi^{-\alpha}}{\partial\bar{x}^2}, \tag{1}$$

where $\Psi=n_0/n$ and n_0, n denote the ion electron densities. Warm-plasma wave breaking will then occur when thermal effects, i.e., pressure, start to dominate over collective effects, inhibiting normal wave advection and destroying the wave [6,8]. This happens when $\Psi\downarrow(\alpha v_T^2/v_\varphi^2)^{1/(\alpha+1)}$, or

$$v=v_\varphi[1-(\alpha v_T^2/v_\varphi^2)^{1/(\alpha+1)}]. \tag{2}$$

In contrast, Ref. [23] presents a family of solutions to Eq. (1) of the form $\Psi(\bar{x},\tau)=1+A(\bar{x})\cos[\omega(\bar{x})\tau]$, and it is claimed that the wave-breaking limit reads $|A(\bar{x})|=1$, i.e., independent of temperature. However, in the derivations of $A(\bar{x})$ and $\omega(\bar{x})$ in Ref. [23], the variable $\theta=\omega(\bar{x})\tau$ is introduced and the operators $\int d\theta$ and $\partial/\partial\bar{x}|_\tau$ are exchanged. This is only allowed if $\omega(\bar{x})$ does not depend significantly on $\bar{x}$. Inserting the above solution $\Psi(\bar{x},\tau)$ into Eq. (1) yields the dispersion-relation

$$\omega^2 = \omega_p^2 - \frac{\alpha v_T^2}{\Psi^{\alpha+1}} \frac{1}{A(\bar{x})} \frac{\partial^2 A}{\partial \bar{x}^2}, \tag{3}$$

so ω depends on $\bar{x}$ through Ψ. Defining $k^2 \equiv -(\partial^2 A/\partial \bar{x}^2)/A$ and $v_\varphi^2 \equiv \omega^2/k^2$, it follows that $\partial\omega/\partial\bar{x}$ can only be neglected when $(\alpha v_T^2/v_\varphi^2)/\Psi^{\alpha+1} < 1$ or $|A(\bar{x})| < 1-(\alpha v_T^2/v_\varphi^2)^{1/(\alpha+1)}$, coincident with the wave-breaking limit [Eq. (2)]. Beyond this limit, the exchange of $\int d\theta$ and $\partial/\partial\bar{x}|_\tau$ is no longer allowed and the solutions presented in Ref. [23] do not actually solve Eq. (1). As a result, the proposed wave-breaking limit $|A(\bar{x})|=1$ derived from them cannot be relied upon.

More generally, any solution to Eq. (1) of the form $\Psi(\bar{x},\tau)=1+k_{\text{eff}}A\cos[k_0\bar{x}-\omega(\bar{x})\tau]$ (where k_0 and k_{eff} denote the initial and effective wave numbers) will lead to dispersion relation (3) and thus, at peak compression, to the relation $v_\varphi^2=\omega_p^2/k^2+\alpha v_T^2/\Psi_{\min}^{\alpha+1} \geq \alpha v_T^2/\Psi_{\min}^{\alpha+1}$, where $\Psi_{\min}$ denotes the minimum value that Ψ obtains during the wave cycle. For a given value of v_φ, it is found that $\Psi_{\min} \geq (\alpha v_T^2/v_\varphi^2)^{1/(\alpha+1)}$. Once again, this corresponds to wave-breaking condition (2), emphasizing its general applicability to both quasistatic and non-quasi-static oscillations. Note that the phase-velocity v_φ cannot vary too much spatially even for non-quasi-static waves, or the wave crests will overtake the troughs and the wave will break rapidly. Also, since the same reasoning can be applied to standing waves $\Psi_0(\bar{x},\tau)=1+kA\cos(k\bar{x})\cos(\omega\tau)$, it follows that there is no fundamental difference between the wave-breaking limits for traveling and standing waves, contrary to claims made in Ref. [23].

Now that the proper wave-breaking limit for non-quasi-static plasma oscillations has been determined, one can proceed to the study of secular behavior. Secular behavior is the phenomenon that the phase difference between neighboring fluid elements in a plasma oscillation is not constant in time. It occurs when the oscillation frequency of the plasma electrons depends on $\bar{x}$ [1,13–16]. This happens, for example, in an inhomogeneous plasma [1,12,13] or for a relativistic plasma oscillation in homogeneous plasma where the oscillation frequency can still depend on $\bar{x}$ through the oscillation amplitude [14,16]. In a cold plasma, secular behavior will cause a plasma oscillation to eventually break at any amplitude. This works as follows. A plasma oscillation having a position-dependent frequency is given by, e.g., $x(\bar{x},\tau)=A\cos[k_0\bar{x}-\omega(\bar{x})\tau]$. The effective wave-number k_{eff} is then derived from $\partial x/\partial\bar{x}$: $k_{\text{eff}}=k_0-(\partial\omega/\partial\bar{x})\tau$, i.e., k_{eff} grows linearly in time. Wave breaking will occur when $|k_{\text{eff}}A|=1$, so even for $k_0A \ll 1$ secular behavior will cause the wave to break after a time of at most $\tau_{\text{WB}}=1/(A|\partial\omega/\partial\bar{x}|)$. Also, by defining the effective phase speed as $v_{\varphi,\text{eff}}=\omega/k_{\text{eff}}$ [13], it follows that wave breaking occurs if the peak-forward fluid speed $v_{\max}$ satisfies $v_{\max}=\omega A=v_{\varphi,\text{eff}}k_{\text{eff}}A=v_{\varphi,\text{eff}}$. Thus, contrary to statements by Lehmann *et al.* [21], the statement that wave breaking happens "when the peak fluid velocity equals the phase speed of the plasma wave" [6] still holds. As shown in Ref. [13], secular behavior will cause the wave's phase speed to decrease until it equals the peak-forward fluid speed, at which point the wave breaks. Electron trapping by a wakefield on a downward density ramp, as demonstrated by Geddes *et al.* [12], is based on this principle.

While the role of secular behavior in cold-plasma wave breaking is well studied [1,13–16], secular behavior in warm-plasma wave breaking is only touched upon by Infeld and Rowlands [24]. Even so, a linearized version of Eq. (1) is used in Ref. [24], which leads to an incorrect wave-breaking limit because it underestimates the plasma pressure. Because of this and because the combination of secular and thermal effects yields some surprising results, this subject will be studied here.

Although thermal effects will normally reduce the wave-breaking amplitude [3–6], they may surprisingly also delay or prevent the onset of wave breaking in the case of secular behavior. This is because secular behavior will make k grow, while thermal effects will make k advect, so the regions of largest k and largest $\partial k/\partial t$ will no longer coincide. The secular growth of k will then saturate eventually, preventing the onset of wave breaking in certain circumstances. As an example, the evolution of k will be investigated in a thermal plasma, on a finite slope where the background density n_0 falls an amount $\Delta n>0$ over a length L, as used, e.g., in electron-trapping experiments by Geddes *et al.* [12]. From $\partial k/\partial\tau+\partial\omega/\partial\bar{x}=0$ [15] and the Bohm-Gross dispersion-relation $\omega^2=\omega_p^2(\bar{x})+\alpha v_T^2k^2$, it is found that (assuming that $k\lambda_D \ll 1$ everywhere)

$$\frac{\partial k}{\partial \tau} + \frac{\alpha v_T^2}{\omega_p} k \frac{\partial k}{\partial \bar{x}} = -\frac{\omega_p}{2n_0}\frac{\partial n}{\partial \bar{x}}. \tag{4}$$

A second "Lagrangianisation" $\tau'=\tau$, $\bar{x}'=\bar{x}-\alpha\lambda_D^2\omega_p\int k\,d\tau'$ [24] yields that $\partial k_{\text{eff}}/\partial\tau'=-(\omega_p/2n_0)\partial n/\partial\bar{x} \approx (\omega_p/2n_0)(\Delta n/L)$. Using $\partial\bar{x}/\partial\tau'=\alpha\lambda_D^2\omega_p k$, this expression integrates to $\Delta(k_{\text{eff}}^2\lambda_D^2) \leq \Delta n/(\alpha n_0)$ over the entire length of the slope, or $k_{\text{eff}}\lambda_D \leq k_{\max}\lambda_D \equiv \sqrt{(k_0\lambda_D)^2+\Delta n/(\alpha n_0)}$, where $k=k_0$ at $\tau=0$. As in Eq. (2), wave breaking then occurs if $k_{\max}A \simeq 1-(\alpha k_{\max}^2\lambda_D^2)^{1/(1+\alpha)}$ (increasing the wave number lowers the wave-breaking limit in two ways). The wave-breaking amplitude A_{WB} and corresponding electric field E_{WB} are then given by

$$A_{\text{WB}} = \frac{\lambda_D\{1-[\alpha(k_0\lambda_D)^2+\Delta n/n_0]^{1/(1+\alpha)}\}}{\sqrt{(k_0\lambda_D)^2+\Delta n/(\alpha n_0)}}, \tag{5}$$

$$E_{\text{WB}} = v_\varphi\{1-C_\alpha[\alpha(k_0\lambda_D)^2+\Delta n/n_0]^{1/(1+\alpha)}\}, \tag{6}$$

where $v_\varphi=\omega(k_{\max})/k_{\max}$ is the local phase speed of the wave [using Bohm-Gross for $\omega(k)$], $C_1 \simeq 1$ [4], and $C_3 \simeq 4/3$ [3]. For a homogeneous plasma, $\Delta n=0$ and $A_{\text{WB}}=[1-(\alpha k_0^2\lambda_D^2)^{1/(1+\alpha)}]/k_0$, a result already known from the work of Dawson [1] and Coffey [3]. For an inhomogeneous cold plasma, $\Delta n>0$ and $\lambda_D=0$, leading to $A_{\text{WB}}=0$. This is a consequence of the fact that the electron oscillations in an inhomogeneous cold plasma exhibit secular behavior [13]: k grows linearly in time, and no matter how small the amplitude, $kA\uparrow 1$ and the wave will eventually break. For both $\Delta n>0$ and $\lambda_D>0$ however, $A_{\text{WB}}>0$ again, provided that $\alpha(k_0\lambda_D)^2+\Delta n/n_0<1$. It follows that the secular behavior that is caused by the plasma inhomogeneity is curbed by thermal effects: k will only grow a finite instead of an unlimited amount, and for sufficiently small A and Δn, wave breaking will not happen in spite of the density ramp.

Equation (5) comes with the following caveat: the Bohm-Gross dispersion relation used to derive it does not incorporate nonlinear effects. In practice, this means that the advection of k will be slightly larger than in the above derivation, while the growth of k due to the density drop Δn will be slightly smaller. As a result, the true value of $A_{\rm WB}$ may be larger than given by Eq. (5). However, most of the nonlinear aspects to warm-plasma wave breaking are still covered, as condition (2) is used for wave breaking rather than $v_{\max} = v_{\varphi,\rm eff}$. Thus, Eq. (5) still provides a reliable leading-order estimate for the wave-breaking limit.

Next, the above results will be extended to also cover plasma oscillations having relativistic amplitudes and/or phase speeds, as encountered in, e.g., laser-wakefield acceleration [12]. Earlier work on wave breaking in warm, relativistic plasma only covered quasistatic waves in homogeneous plasma, to the exclusion of secular behavior [5,6]. The Lagrangian equation for oscillations in a warm relativistic plasma can be derived from the fully relativistic model of Katsouleas and Mori (K&M) [5],

$$\frac{\partial}{\partial t}[(U+P)\gamma^2 v] + \frac{\partial}{\partial x}[c^2 P + (U+P)\gamma^2 v^2] + (e/m_e)nE = 0,$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)E = \frac{n_0 e}{\varepsilon_0}v,$$

$$\frac{\partial n}{\partial t} + \frac{\partial nv}{\partial x} = 0.$$

Here, v denotes the mean plasma velocity, $\gamma = 1/\sqrt{1-v^2}$, and U and P are the internal energy and pressure of the electrons in the comoving rest frame, scaled with $m_e c^2$. Note that the average momentum p will not be used for the moment, since $p \neq v/\sqrt{1-v^2}$ in a relativistic thermal plasma.

Using K&M's expressions for U and P,

$$\begin{Bmatrix} P \\ U \end{Bmatrix} = \frac{n_0}{2p_0}[n_p p_0\sqrt{1+(n_p p_0)^2} \mp \sinh^{-1}(n_p p_0)],$$

where $p_0^2 = \beta \equiv 3k_B T/(m_e c^2)$ and $n_p = (n/n_0)/\gamma$; the above equations can be manipulated into

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)[\gamma v\sqrt{1+(n_p p_0)^2}] + \frac{c^2}{n}\frac{\partial P}{\partial x} + (e/m_e)E = 0.$$

This can directly be transformed into Lagrangian coordinates $(\bar{x},\tau)$,

$$\frac{\partial}{\partial \tau}[\gamma v\sqrt{1+(n_p p_0)^2}] + \frac{c^2}{n_0}\frac{\partial P}{\partial \bar{x}} + \omega_p^2(x-\bar{x}) = 0.$$

Note that for $p_0 = 0$ (cold plasma), this equation reduces to the one found by Polovin [25]: $\partial^2 p/\partial\tau^2 + \omega_p^2 p/\sqrt{1+p^2} = 0$.

While a complete treatment of this fully relativistic equation is beyond the scope of this paper, one can still investigate the weakly relativistic case $\beta/(1-v_\varphi^2) \ll 1$, where v_φ denotes the wave's phase speed. First, the case of an oscillation with constant amplitude in an inhomogeneous plasma is considered: $p = p_m\cos(k\bar{x} - \omega\tau)$, with p the average momentum. For fully relativistic waves, i.e., $p_m \gtrsim 1$, wave breaking occurs when $p_m \sim v_\varphi/\sqrt{1-v_\varphi^2}$, and such waves can persist mostly by virtue of having a phase speed satisfying $\gamma_\varphi^2 \equiv 1/(1-v_\varphi^2) \gg 1$. A small drop in v_φ due to secular behavior will lead to a large drop in γ_φ, and a fully relativistic wave will then break within a couple of oscillations, regardless of the presence or absence of thermal effects. The influence of thermal effects on secular behavior will be more visible for weakly relativistic waves, $|p_m| \ll 1$, so this case will be studied here.

For $|p_m| \ll 1$, the following approximations can be used, $\gamma v \approx p$, $P \approx \beta n_0 n_p^3/3$, $\partial P/\partial\bar{x} \approx \beta n_0 k^2(x-\bar{x})$, and $\sqrt{1+(n_p p_0)^2} \approx \sqrt{1+\beta}$, leading to the following dispersion–relation,

$$\omega^2(1+\beta) = \omega_p^2(\bar{x})[1-3p_m^2/8] + \beta c^2 k^2. \qquad (7)$$

As before, this dispersion relation can be turned into an advection equation for the wave number k,

$$\frac{\partial k}{\partial \tau} + \frac{\alpha v_T^2}{\omega_p(1+\beta)}k\frac{\partial k}{\partial \bar{x}} = -\frac{\omega_p}{2n_0}\frac{1-3p_m^2/16}{1+\beta}\frac{\partial n}{\partial \bar{x}}.$$

Note the similarity to the nonrelativistic equation (4). Once again, this equation is applied to the example of slope where the background density n_0 falls an amount $\Delta n > 0$ over a length L. As in the nonrelativistic case, it is found that wave breaking due to secular behavior is curbed by thermal effects below a certain wave-amplitude $E_{\rm WB}$. This limiting amplitude can be found by replacing v_T^2 by $v_T^2/(1+\beta)$ and Δn by $\Delta n(1-3p_m^2/8)/(1+\beta)$ in Eqs. (5) and (6).

The above results are particularly relevant to wakefield acceleration on a downward plasma-density ramp [9–12]. For example, when applying these results to the experiment by Geddes *et al.* [12], it is found that their wakefield amplitude is well above the wave-breaking threshold $E_{\rm WB}$, mainly because they use a very tightly focused laser pulse $[eE_l/(m_e\omega_l c) \sim 2]$ and because $\Delta n/n_0$ is large at the bottom of the ramp where $n_0 \downarrow 0$. However, if this experiment were to be repeated using a wider laser focus to trap more electrons and a constant density plateau at the bottom of the ramp for further acceleration (so $\Delta n/n_0$ remains smaller), then thermal effects resulting from plasma heating by the laser prepulse may be of significant influence on the electron-trapping process on a downward density ramp.

As shown by Drake *et al.* [14], secular behavior also occurs for relativistic waves in a homogeneous plasma, as long as the wave amplitude is position dependent. Because dispersion relation (7) contains the wave's amplitude, a position-dependent amplitude will lead to a position-dependent frequency, leading to secular behavior even for a homogeneous background plasma. For example, for a small-amplitude oscillation with $\bar{x}$-dependent amplitude, $p = p_m\cos(k\bar{x})\cos(\omega\tau)$ with $|p_m| \ll 1$, the dispersion relation becomes

$$\omega^2(1+\beta) = \omega_p^2[1-3p_m^2\cos^2(k\bar{x})/8] + \beta c^2 k^2. \qquad (8)$$

Using the same techniques as in the case of an inhomogeneous plasma, it follows once more that the secular growth of k is curbed by thermal effects

$$k^2 \leq k_{\max}^2 \equiv k_0^2 + \frac{3}{8}\frac{p_m^2}{\beta}\frac{\omega_p^2}{c^2}.$$

For weakly relativistic waves, wave breaking sets in when the mean plasma velocity satisfies $v \approx v_\varphi[1 - (\gamma_\varphi^2 \beta / v_\varphi^2)^{1/4} / \gamma_\varphi^2]$, where $v_\varphi = \omega / k$ and $\gamma_\varphi = 1/\sqrt{1 - v_\varphi^2}$. Using $v_\varphi = \omega_{\max}/k_{\max}$, $\omega_{\max} \equiv \omega(k_{\max}) \approx [\omega_p^2(1 - 3p_m^2/16) + \beta c^2 k_{\max}^2]^{1/2}$, and assuming that secular behavior is dominant, i.e., $p_m^2 \omega_p^2 / (c^2 k_0^2 \beta) \gg 1$, so $k_m \approx \sqrt{3/(8\beta)} p_m \omega_p / c$, it follows that, at wave breaking, p_m satisfies

$$\frac{c}{v_\varphi}\frac{p_m}{\sqrt{1+p_m^2}} = 1 - (1 - v_\varphi^2)[(\beta/v_\varphi^2)/(1 - v_\varphi^2)]^{1/4}. \quad (9)$$

To leading approximation, this returns $p_m \approx (8\beta/3)^{1/4}$ as the amplitude limit below which relativistic secular behavior will be curbed by thermal effects. The corresponding (scaled) electric field is given by $E_{\mathrm{WB}} \approx v_\varphi / c = (8\beta/3)^{1/4}$. This is again in contradistinction to the secular behavior in a cold plasma [14], where the unlimited growth of k will always slow the wave down until it breaks, however small the amplitude p_m.

In summary, wave breaking has been studied for general, non-quasi-static plasma waves. It has been found that breaking of such waves has many traits in common with the breaking of quasistatic waves, confirming the notion that quasistatic wave breaking is a proper special case of general wave breaking. A proper investigation of secular behavior in thermal plasma has been carried out here for the first time. This investigation includes relativistic effects and the first ever derivation of the Lagrangian equation for a fully relativistic, non-quasi-static, thermal plasma. It has been found that thermal effects can curb secular behavior and prevent wave breaking in certain specific circumstances, even though thermal effects normally facilitate wave breaking [3–6]. The wave-breaking limits for plasma waves in inhomogeneous thermal plasma and for inhomogeneous relativistic waves in homogeneous plasma have been derived for the first time. As such, this work is a generalization of earlier work on breaking of quasistatic waves in thermal plasma [6], with important consequences for the study of inhomogeneous plasma oscillations or plasma waves in inhomogeneous plasma. This has particular relevance in light of recent experiments on electron trapping on a downward plasma density ramp, which yielded electron bunches having a very small absolute energy spread with great potential for further acceleration in a two-stage scheme [12]. Using the models developed here, electron trapping by a plasma wave on a downward density ramp can be tuned to reach conditions that are optimal for further acceleration of the trapped electron bunch.

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